

Physics 1C Discussion (Week 3)

Problem 1

A circular conducting ring with radius $r_0 = 0.042$ m lies in the xy -plane in a region of uniform magnetic field

$$\vec{B} = B_0 \left[1 - 3 \left(\frac{t}{t_0} \right)^2 + 2 \left(\frac{t}{t_0} \right)^3 \right] \hat{k}, \quad (1)$$

where $t_0 = 0.0100$ s $= 1.0 \times 10^{-2}$ s and is constant, t is time, $\hat{k}$ is the unit vector in the $+z$ -direction, and $B_0 = 0.0800$ T and is constant. At points a and b , there is a small gap in the ring with wires leading to an external circuit of resistance $R = 12.0 \Omega$. There is no magnetic field at the location of the external circuit.

(a) Derive an expression, as a function of time, for the total magnetic flux Φ_B through the ring.

First, the normal vector $\hat{n}$ of the surface enclosed by the loop of this circular wire is $\hat{k}$. By definition, you get the total magnetic flux passing through the ring in $\hat{n}$ direction by

$$\begin{aligned} \Phi_B &= \int_S dS \hat{n} \cdot \vec{B}(t) \\ &= \int_S dS \hat{k} \cdot \left\{ B_0 \left[1 - 3 \left(\frac{t}{t_0} \right)^2 + 2 \left(\frac{t}{t_0} \right)^3 \right] \hat{k} \right\} \\ &= B_0 \left[1 - 3 \left(\frac{t}{t_0} \right)^2 + 2 \left(\frac{t}{t_0} \right)^3 \right] \int_S dS \hat{k} \cdot \hat{k} \\ &= B_0 \left[1 - 3 \left(\frac{t}{t_0} \right)^2 + 2 \left(\frac{t}{t_0} \right)^3 \right] \int_S dS \\ &= (\pi r_0^2) B_0 \left[1 - 3 \left(\frac{t}{t_0} \right)^2 + 2 \left(\frac{t}{t_0} \right)^3 \right]. \end{aligned} \quad (2)$$

(b) Determine the *emf* (electromotive force) induced in the ring at time $t = 5.00 \times 10^{-3}$ s. What is the polarity of the emf?

$$\begin{aligned} \mathcal{E}(t) &= -\frac{d\Phi_B}{dt} \\ &= -(\pi r_0^2) B_0 \left[0 - 6 \left(\frac{t}{t_0^2} \right) + 6 \left(\frac{t^2}{t_0^3} \right) \right] \\ &= -6 \left(\frac{\pi r_0^2}{t_0} \right) B_0 \left[\left(\frac{t}{t_0} \right)^2 - \left(\frac{t}{t_0} \right) \right]. \end{aligned} \quad (3)$$

Then at time $t = 5.00 \times 10^{-3}$ s, you get

$$\begin{aligned} \mathcal{E}(t) &= -6 \left(\frac{\pi r_0^2}{t_0} \right) B_0 \left[\left(\frac{t}{t_0} \right)^2 - \left(\frac{t}{t_0} \right) \right] \\ &= \frac{3}{2} \left(\frac{\pi r_0^2}{t_0} \right) B_0 \\ &\simeq 6.6 \times 10^{-2} \text{ V}. \end{aligned} \quad (4)$$

Therefore, the polarity is given by

$$\begin{aligned} I &= \frac{\mathcal{E}(t)}{R} \\ &\simeq 5.5 \times 10^{-3} \text{ A}, \end{aligned} \quad (5)$$

which goes anticlockwise from point b to a . Or equivalently, the voltage at a is higher than that of b .

(c) Because of the internal resistance of the ring, the current through R at the time given in part (b) is only $3.00 \text{ mA} = 3.0 \times 10^{-3} \text{ A}$. Determine the internal resistance of the ring.

Let r be the internal resistance of the ring, then the Ohm's law reads

$$\begin{aligned} R + r &= \frac{\mathcal{E}(t)}{I_{\text{measured}}} \\ &\simeq \frac{6.6 \times 10^{-2} \text{ V}}{3.0 \times 10^{-3} \text{ A}} \\ &= 22 \Omega. \end{aligned} \tag{6}$$

Then, the internal resistance r is given by

$$\begin{aligned} r &= 22 \Omega - R \\ &= 22 \Omega - 12 \Omega \\ &= 10 \Omega. \end{aligned} \tag{7}$$

(d) Determine the emf in the ring at a time $t = 1.21 \times 10^{-2} \text{ s}$. What is the polarity of the emf?

At time $t = 1.21 \times 10^{-2} \text{ s}$, you get

$$\begin{aligned} \mathcal{E}(t) &= -6 \left(\frac{\pi r_0^2}{t_0} \right) B_0 \left[\left(\frac{t}{t_0} \right)^2 - \left(\frac{t}{t_0} \right) \right] \\ &\sim -152 (\pi r_0^2) B_0 \\ &\simeq -6.6 \times 10^{-2} \text{ V}. \end{aligned} \tag{8}$$

Therefore, the polarity is given by

$$\begin{aligned} I &= \frac{\mathcal{E}(t)}{R + r} \\ &= -3.0 \times 10^{-3} \text{ A}, \end{aligned} \tag{9}$$

which goes clockwise from point a to b . Or equivalently, the voltage at b is higher than that of a .

(e) Determine the time at which the current through R reverses its direction.

The current direction reverses when $d\Phi_B/dt = 0$. Let the reversing time as t_{turn} , then the reversing condition is written as

$$\begin{aligned} \frac{d\Phi_B}{dt}(t_{\text{turn}}) &= 6 \left(\frac{\pi r_0^2}{t_0} \right) B_0 \left[\left(\frac{t_{\text{turn}}}{t_0} \right)^2 - \left(\frac{t_{\text{turn}}}{t_0} \right) \right] = 0 \\ &\Rightarrow \left(\frac{t_{\text{turn}}}{t_0} \right)^2 - \left(\frac{t_{\text{turn}}}{t_0} \right) = 0 \\ &\quad \left(\frac{t_{\text{turn}}}{t_0} \right) \left[\left(\frac{t_{\text{turn}}}{t_0} \right) - 1 \right] = 0. \end{aligned} \tag{10}$$

Since $t_{\text{turn}} > 0$, you get

$$\begin{aligned} \left(\frac{t_{\text{turn}}}{t_0} \right) - 1 &= 0 \\ \therefore t_{\text{turn}} &= t_0 = 1.0 \times 10^{-2} \text{ s} \end{aligned} \tag{11}$$